LAPLACE TRANSFORMS AND DIFFERENTIAL EQUATIONS

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Laplace transforms can be used to solve differential equations. The Laplace transform for a derivative is given in Saff and Snider's equation 8 in section 8.3.

$$\mathcal{L}\left\{F^{(k)}\right\}(s)=s^{k}\mathcal{L}\{F\}(s)-s^{k-1}F(0)-s^{k-2}F'(0)-\ldots-F^{(k-1)}(0) \tag{1}$$

where $F^{(k)}$ is the kth derivative. To keep the notation uncluttered we'll use a table of Laplace transforms instead of working them out from scratch in each case. A table can be found in Saff and Snider, Chapter 8, or on Wikipedia.

The idea is to transform both sides of a differential equation and then solve for the Laplace transform of the function, then invert the transform to get the function itself.

Example 1. Solve

$$f'-f=e^{3t} \tag{2}$$

with

$$f(0)=3 \tag{3}$$

From 1 we have, with $L\equiv\mathcal{L}\{f\}(s)$

$$\begin{aligned}
sL-f(0)-L&=\mathcal{L}\left(e^{3t}\right) &(4)\\
sL-3-L&=\frac{1}{s-3} &(5)\\
L&=\frac{1}{s-1}\left(\frac{1}{s-3}+3\right) &(6)\\
&=\frac{3s-8}{(s-1)(s-3)} &(7)\\
&=\frac{5}{2(s-1)}+\frac{1}{2(s-3)} &(8)
\end{aligned}$$

where we used partial fractions to get the last line. From tables, we can invert the transform to get

$$f(t)=\frac{5}{2}e^{t}+\frac{1}{2}e^{3t} \tag{9}$$

Example 2. Solve

$$f''-5f'+6f=0 \tag{10}$$

with

$$\begin{aligned} f(0)&=1\\ f'(0)&=-1 \end{aligned} \tag{11}$$

We have

$$s^{2}L-sf(0)-f'(0)-5(sL-f(0))+6L=0 \tag{12}$$

$$s^{2}L-s+1-5(sL-1)+6L=0 \tag{13}$$

$$L=\frac{s-6}{s^{2}-5s+6} \tag{14}$$

$$=\frac{4}{s-2}-\frac{3}{s-3} \tag{15}$$

where we again used partial fractions in the last line. Inverting, we get

$$f(t)=4e^{2t}-3e^{3t} \tag{16}$$

Example 3. Solve

$$f''-f'-2f=e^{-t}\sin 2t \tag{17}$$

with

$$\begin{aligned} f(0)&=0\\ f'(0)&=2 \end{aligned} \tag{18}$$

The transform of the LHS is

$$s^{2}L-sf(0)-f'(0)-(sL-f(0))-2L=s^{2}L-2-sL-2L \tag{19}$$

The transform of the RHS is, from tables

$$\mathcal{L}\left\{e^{-t}\sin 2t\right\}=\frac{2}{(s+1)^{2}+4} \tag{20}$$

Combining the two we get

$$L=\frac{1}{s^{2}-s-2}\left[\frac{2}{(s+1)^{2}+4}+2\right] \tag{21}$$

$$= \frac{2(s+1)^2+10}{(s^2-s-2)\left[(s+1)^2+4\right]} \tag{22}$$

$$= \frac{2s^2+4s+12}{(s+1)(s-2)\left[(s+1)^2+4\right]} \tag{23}$$

$$= \frac{28}{39(s-2)} - \frac{5}{6(s+1)} + \frac{3s-1}{26\left[(s+1)^2+4\right]} \tag{24}$$

Using tables, we find

$$f(t) = \frac{28}{39}e^{2t} - \frac{5}{6}e^{-t} + \frac{3}{26}e^{-t}\cos 2t - \frac{2}{26}e^{-t}\sin 2t \tag{25}$$

We can verify that this is a solution by substituting into 17, although it's a bit tedious.

Example 4. Solve

$$f'' - 3f' + 2f = \begin{cases} 0 & 0 \le t < 3 \\ 1 & 3 \le t \le 6 \\ 0 & t > 6 \end{cases} \tag{26}$$

with

$$f(0) = f'(0) = 0 \tag{27}$$

The transform of the LHS is

$$s^2L - 3sL + 2L = \left(s^2 - 3s + 2\right)L \tag{28}$$

The transform of the RHS is

$$\frac{e^{-3s} - e^{-6s}}{s} \tag{29}$$

so we have

$$L = \frac{e^{-3s} - e^{-6s}}{s(s^2 - 3s + 2)} \tag{30}$$

To invert the transform, we make use the fact that the transform of a delayed function is, for a function

$$f_\tau(t) \equiv \begin{cases} 0 & 0 \le t < \tau \\ f(t-\tau) & \tau \le t < \infty \end{cases} \tag{31}$$

given by

$$\mathcal{L}\{f_\tau(u)\} = e^{-s\tau}\mathcal{L}\{f(u)\} \tag{32}$$

We begin by splitting 30 into partial fractions

$$L = \left(e^{-3s} - e^{-6s}\right)\left[\frac{1}{s(s-1)(s-2)}\right] \tag{33}$$

$$= \left(e^{-3s} - e^{-6s}\right)\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right] \tag{34}$$

The inverse transform of the terms in brackets is

$$L^{-1}\left[\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right] = \frac{1}{2} - e^{t} + \frac{1}{2}e^{2t} \tag{35}$$

From 32 we get

$$L^{-1}\left[e^{-3s}\left(\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right)\right] = H(t-3)\left(\frac{1}{2} - e^{t-3} + \frac{1}{2}e^{2(t-3)}\right) \tag{36}$$

$$L^{-1}\left[e^{-6s}\left(\frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2(s-2)}\right)\right] = H(t-6)\left(\frac{1}{2} - e^{t-6} + \frac{1}{2}e^{2(t-6)}\right) \tag{37}$$

where $H(t)$ is the Heaviside, or step, function

$$H(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{38}$$

The solution is then

$$f(t) = \begin{cases} 0 & t < 3 \\ \frac{1}{2} - e^{t-3} + \frac{1}{2}e^{2(t-3)} & 3 \le t \le 6 \\ \frac{1}{2} - e^{t-3} + \frac{1}{2}e^{2(t-3)} - \left(\frac{1}{2} - e^{t-6} + \frac{1}{2}e^{2(t-6)}\right) & t > 6 \end{cases} \tag{39}$$

We can verify this by substituting into 26.